Circuit method for conductive and ferromagnetic materials

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***Abstract*— To find the induced currents in conductive objects, the circuit method (CM) replaces these objects by a set of magnetically (mutually) coupled filaments. The unknown induced currents are obtained by solving an electrical circuit and the magnetic field is found by the Biot-Savart law. In the literature, the method is described as a fast and easy-to-implement alternative for the finite element method (FEM), with however some limitations: the objects to model in the CM should be non-ferromagnetic and much tinner than the penetration depth. In this paper, the CM is extended. It also models ferromagnetic behaviour and shields thicker than the penetration depth, by adding extra filaments carrying proper currents. To validate the CM, an axisymmetric shielding problem is solved both with CM and FEM for several conductive and/or ferromagnetic shields.**

I. INTRODUCTION

The principle of the circuit method (CM) is the replacement of conductive regions by a set of mutually coupled filaments. In the axisymmetric case, the filaments are coils. Similar to the finite element method (FEM), the CM becomes more accurate and more computationally demanding when more filaments are taken and the grid or mesh becomes denser. However, for a variety of problems with complex geometries in 3D, the CM is often much faster than the FEM, especially for long, thin objects such as shields in magnetic screening applications. Moreover, the CM is easy to implement. The CM described in literature [1], [2] has the disadvantages that it can't model ferromagnetic objects nor objects thicker than the penetration depth. This paper tries to cope with these problems.

II. THE CIRCUIT MODEL

In the classical circuit model for **conductive objects**, the objects that carry induced currents are replaced by a set of equivalent conducting filaments. We choose an axisymmetric geometry so that the filaments are coaxial coils. For example in Fig. 1a, the conductive object is an axisymmetric shield that has to reduce the magnetic field created by the excitation coil with current I_{e}. The conductive shield is replaced by a set of P coils with currents $I_{\mathrm{c}k}$, $k = 1 \ldots P$. Each coil k has its own resistance $R_{\mathrm{c}k}$, self inductance L_k and mutual inductances M_{ki} between the coil k and other coils i. This results in an electrical circuit shown in Fig. 1b. If the permeable shield and the corresponding branches with indices l and r are ignored, the circuit of Fig. 1b is described by

$$[-\mathbf{U}_{\mathrm{c}}] = [Z_{\mathrm{cc}}][\mathbf{I}_{\mathrm{c}}] \tag{1}$$

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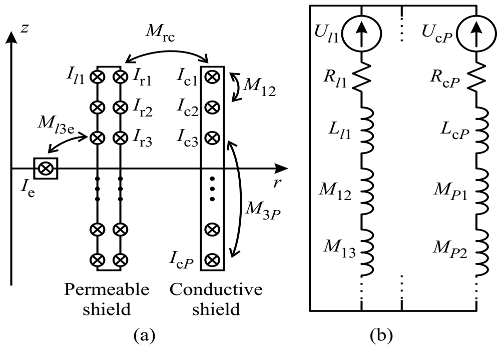

Fig. 1. In the studied shielding application, the shields are replaced by a set of mutually coupled one-turn coils in (a), resulting in the electrical circuit (b). The mutual coupling of shield coil k with the excitation coil is replaced by a voltage source U_k. The figure is not on scale; the radius r, thickness d and height h of the shields are in Table I; the excitation coil is at r = 0.2 m.

with $\mathbf{I}_{\mathrm{c}}$ the array with currents and $[Z_{\mathrm{cc}}]$ the impedance matrix

$$[Z_{\mathrm{cc}}] = \begin{bmatrix} R_{\mathrm{c}1} + \mathrm{j}\omega L_{\mathrm{c}1} & \mathrm{j}\omega M_{12} & \ldots & \mathrm{j}\omega M_{1P} \\ \mathrm{j}\omega M_{21} & R_{\mathrm{c}2} + \mathrm{j}\omega L_{\mathrm{c}2} & \ldots & \mathrm{j}\omega M_{2P} \\ \vdots & \vdots & \ddots & \vdots \\ \mathrm{j}\omega M_{P1} & \mathrm{j}\omega M_{P2} & \ldots & R_{\mathrm{c}P} + \mathrm{j}\omega L_{\mathrm{cP}} \end{bmatrix}$$

wherein ω is the angular velocity. The mutual couplings with the source current I_{e} are represented by the array of voltage sources $\mathbf{U}_{\mathrm{c}} = [U_{\mathrm{c}1} \; U_{\mathrm{c}2} \; \ldots \; U_{\mathrm{c}P}]$.

The components in Fig. 1b are calculated by well-known analytical formula's for non-ferromagnetic coils. The *resistance* of a coil is the resistance of a circular conductor with the same resistivity as the material it replaces and with cross section chosen such that the section of all coils together is equal to the cross section of the object. The expressions for *self inductance* and *mutual inductances* between coils are found in [3]. With all components known, the array $\mathbf{I}_{\mathrm{c}}$ with the coil currents is the solution of (1). The magnetic field of a coil with given radius and current is given by Biot-Savart's law.

For **ferromagnetic (and conductive) objects**, the analytical formula's to calculate the inductances are not valid. However, the ferromagnetic behaviour can be simulated by introducing equivalent currents. Indeed, if the law of Ampere $\nabla \times \mathbf{H} = \mathbf{J}$ is combined with the constitutive law $\mathbf{B} = \mu_0 \mathbf{H} + \mu_0 \mathbf{M}$, we obtain $\nabla \times \frac{\mathbf{B}}{\mu_0} = \mathbf{J} + \nabla \times \mathbf{M}$. It is clear that we have to add the current density $\mathbf{J}_{\mathrm{p}} = \nabla \times \mathbf{M}$ in order to represent the ferromagnetic object. In an axisymmetric problem with cylindrical coordinates (r, ϕ, z), $\mathbf{J}_{\mathrm{p}}$ has only a ϕ-component:

$$\mathbf{J}_{\mathrm{p}} = J_{\mathrm{p}} \; \mathbf{1}_{\phi} = \left(\frac{\partial M_r}{\partial z} - \frac{\partial M_z}{\partial r} \right) \mathbf{1}_{\phi} \tag{2}$$

In many cases, the field is mainly along one direction, e.g. along the z-direction in the shielding problem of Fig. 1 studied

TABLE I

PROPERTIES AND CPU TIMES OF THE SIMULATED SHIELDS

Shield	r [mm]	d [mm]	h [mm]	σ [M℧]	μ_r	Coils	CPU-time [s] CM	FEM
Cond.	300	4.0	200	52	1	1×25	0.49	54.89
Cond.	300	4.0	200	52	1	2×25	1.38	54.89
Perm.	300	5.0	100	0	100	2×4	0.09	23.41
Perm.	300	5.0	100	0	100	2×16	0.38	23.41
Double	300	5.0	100	0	100	2×25	1.42	40.90
	320	1.0	100	52	1	1×25		

in the next section. In this case, the current to be added along a vertical boundary of the shield is

$$J_{\mathrm{p}} = -\frac{\partial M_z}{\partial r} = -\frac{M_z^+ - M_z^-}{r^+ - r^-} \tag{3}$$

where the notations (+) and (-) symbolize the right and left side of a boundary of the ferromagnetic sheet. For the left boundary, $M_z^- = 0$ and $M_z^+ = (\mu_r - 1)\ H_z$ are the magnetization in air and in the linear ferromagnetic shield. Eq. (3) expresses that the flux density at the shield (+) side of the left boundary is μ_r times the flux density at the air (-) side of this boundary. If (3) is combined with (1), the unknown currents $\mathbf{I}_\mathrm{l}$ and $\mathbf{I}_\mathrm{r}$ (see Fig. 1) to model ferromagnetism and the unknown currents $\mathbf{I}_\mathrm{c}$ to model conductivity are found from:

$$\begin{bmatrix} \mathbf{h}_{\mathrm{ll}}^+ - \mu_\mathrm{r}\mathbf{h}_{\mathrm{ll}}^- & \mathbf{h}_{\mathrm{lr}}^+ - \mu_\mathrm{r}\mathbf{h}_{\mathrm{lr}}^- & \mathbf{h}_{\mathrm{lc}}^+ - \mu_\mathrm{r}\mathbf{h}_{\mathrm{lc}}^- \\ \mathbf{h}_{\mathrm{rl}}^- - \mu_\mathrm{r}\mathbf{h}_{\mathrm{rl}}^+ & \mathbf{h}_{\mathrm{rr}}^- - \mu_\mathrm{r}\mathbf{h}_{\mathrm{rr}}^+ & \mathbf{h}_{\mathrm{rc}}^- - \mu_\mathrm{r}\mathbf{h}_{\mathrm{rc}}^+ \\ \mathrm{j}\omega[M_{\mathrm{cl}}] & \mathrm{j}\omega[M_{\mathrm{cr}}] & [Z_{\mathrm{cc}}] \end{bmatrix} \begin{bmatrix} \mathbf{I}_\mathrm{l} \\ \mathbf{I}_\mathrm{r} \\ \mathbf{I}_\mathrm{c} \end{bmatrix} = \begin{bmatrix} \mathbf{U}_\mathrm{l} \\ \mathbf{U}_\mathrm{r} \\ -\mathbf{U}_\mathrm{c} \end{bmatrix} \tag{4}$$

where $\mathbf{h}_{ij}$ (i, j = l, r or c) are matrices containing the z-components of the field in points at boundary i caused by 1 A current in a conductor at boundary j. The fields due to the excitation current I_e cause at the left and right boundaries the fields $\mathbf{H}_{\mathrm{el}}$ and $\mathbf{H}_{\mathrm{er}}$, resulting in the equivalent sources $\mathbf{U}_\mathrm{l} = \mu_\mathrm{r}\mathbf{H}_{\mathrm{el}}^- - \mathbf{H}_{\mathrm{el}}^+$ and $\mathbf{U}_\mathrm{r} = \mu_\mathrm{r}\mathbf{H}_{\mathrm{er}}^+ - \mathbf{H}_{\mathrm{er}}^-$ resp. in Fig. 1b.

III. APPLICATION OF THE CM TO A SHIELDING PROBLEM

We consider the axisymmetric shielding problem of Fig. 1 with several types of shields. The properties of the shields – radius r, thickness d, height h, conductivity σ, permeability μ_r and numbers of equivalent coils – are in Table I.

Conductive shield The thickness of the conductive shield is larger than the penetration depth which is 2.21 mm at the chosen frequency of 1 kHz. The excitation current at $r = 0.2$ m is 40 A. Simulations show that 1 layer of 25 coils approximates the FEM curve quite well without modelling skin effect. With 2 layers of 25 conductors, the CM is still fast (Table I) and more accurate because skin effect is taken into account.

Ferromagnetic shield With a non-conductive, ferromagnetic shield, Fig. 2 shows how many equivalent conductor pairs P should be chosen to obtain accurate results. Fig. 3 illustrates the corresponding field distribution. For two pairs, the shield is not well modelled: the resulting field is as high as the field without shield. With seven pairs, the correspondence with the FEM-curve is good except very close to the shield (in the region $r < 0.35$ m). Sixteen conductor pairs give rise to an accurate approximation of the FEM curve. The field pattern with 16 equivalent conductor pairs (Fig. 3, right plot) can hardly be distinguished from the one of a ferromagnetic shield.

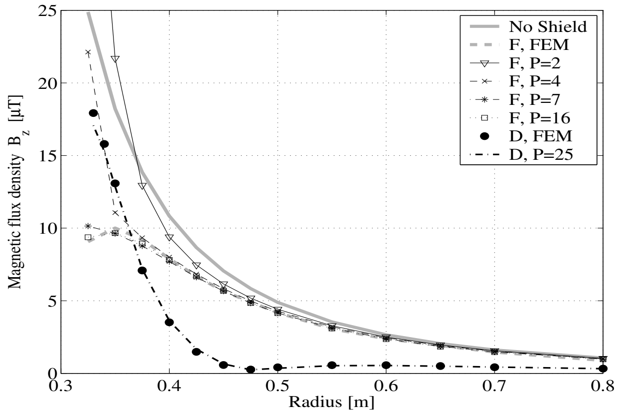

Fig. 2. Amplitude of the magnetic flux density in the z = 0 plane for a ferromagnetic shield (F) or a double ferromagnetic and conductive shield (D) with properties in Table 1. The FEM is compared with the CM for which the number of rows of equivalent coils is given in the legend.

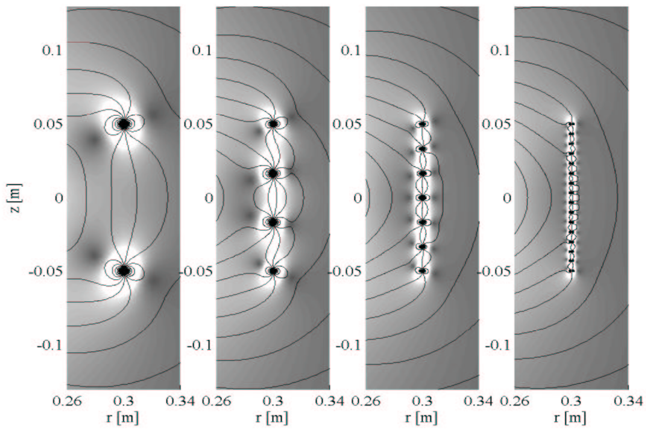

Fig. 3. Field patterns in the region of the ferromagnetic shield that is replaced by P = 2, 4, 7 or 16 equivalent conductor pairs respectively

Double shield If both a conductive and ferromagnetic shield are present, or if one shield has both high σ and high μ, all currents $\mathbf{I}_\mathrm{l}$, $\mathbf{I}_\mathrm{r}$ and $\mathbf{I}_\mathrm{c}$ are found from (4). The number of equivalent conductors is 75: two layers of 25 coils for the ferromagnetic and one layer of 25 for the conductive shield. Fig. 2 shows that with P = 25, enough accuracy is obtained, while the CPU time is only 1.42 s – see Table I.

IV. CONCLUSION

The circuit method is used to model ferromagnetic and/or conductive objects. The ferromagnetic behaviour is taken into account by adding a suitable current distribution. In the axisymmetric shielding application, the extra current distribution consists of two layers of coaxial coils. The CM achieves good accuracy and is on average 30 times faster than FEM.

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